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Minimality of totally geodesic submanifolds in Finsler geometry

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Abstract Using the symplectic definition of the Holmes-Thompson volume we prove that totally geodesic submanifolds of a Finsler manifold are minimal for this volume. Thanks to well suited technics the minimality of totally geodesic hypersurfaces (see Álvarez Paiva and Berck in Adv Math 204(2):647–663, 2006) and 2-dimensional totally geodesic surfaces (see Álvarez Paiva and Berck in Adv Math 204(2):647–663, 2006, Ivanov in Algebra i Analiz 13(1)26–38, 2001) had already been proved. However the corresponding statement for the Hausdorff measure is known to be wrong even in the simplest case of totally geodesic 2-dimensional surfaces in a 3-dimensional Finsler manifold (see Álvarez Paiva and Berck in Adv Math 204(2):647–663, 2006).

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1 Introduction

Starting to study the volumes and areas in Finsler manifolds everyone may be puzzled by the existence of several natural volume definitions. The two most studied being the Hausdorff measure, strongly supported by Busemann, and the Holmes-Thompson volume which, thanks to its close connections with convex and symplectic geometries, more and more appears as the appropriate notion for Finsler manifolds. It is also the adequate definition to extend to Finsler geometry many classical results of integral geometry such as Crofton formulae (see [3–5]).

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Nevertheless, despite the growing attention paid to these volumes and many interesting results obtained, the study of the variational problem, and more precisely of minimal submanifolds, is still at its very beginning (see [7,9,11,16,17]). Our main theorem, extending to Finsler manifolds a classical result of Riemannian geometry, is a first characterization of a family of minimal submanifolds.

Theorem *Totally geodesic submanifolds of Finsler manifolds are minimal for the Holmes-Thompson volume.*

This result also extends to all dimensions partial results obtained thanks to well suited technics. Let's mention as examples that the minimality of 2-dimensional totally geodesic surfaces is a consequence of Ivanov's filling theorem (see [7,15]), while in [7] we proved with J.C.Álvarez the minimality of totally geodesic hypersurfaces using a generalized Crofton formula. Finally the minimality of projective subspaces in projective Finsler manifolds, i.e. for which all the geodesic are straight lines, is a consequence of results obtained by Álvarez and Fernandes in [4] also using Crofton formulae.

All these results concern the Holmes-Thompson volume. Oppositely explicit counter-examples were given in [7] for the Hausdorff measure, even in the simplest case of 2-dimensional totally geodesic surfaces in a 3-dimensional Finsler manifold. But since this statement is a very classical result in Riemannian geometry where being totally geodesic is equivalent to the vanishing of the second fundamental form (see [12]), one may view it as a natural requirement for a 'good' notion of volume, arguing also for the preponderance of the Holmes-Thompson volume for Finsler manifolds over other notions.

This theorem is a consequence of the symplectic nature of the Holmes-Thompson volume. Therefore we will not use its original definition (see [14]) but the following which furthermore tends to become the standard one:

Definition The volume of a n-dimensional Finsler manifold is the Liouville volume of its unit co-sphere bundle divided by the Euclidean volume of the $n-1$-dimensional Euclidean unit sphere.

Of course the volume of a submanifold is defined in the same way as a multiple of the volume of its own unit co-sphere bundle for the induced metric. However to characterize totally geodesic submanifolds and to compare the volume of close submanifolds it will be worthwhile not to consider their co-sphere bundles on their own but to embed them into the co-sphere bundle of the ambient manifold. This will be done using the Legendre map and we will call the image of such an embedding the *cotangent lift* of the submanifold.

Using this we will then proof in Proposition 6 that a submanifold is totally geodesic provide the Hamiltonian field is tangent to its cotangent lift. Also we will give in Theorem 1 a simple and elegant characterization of minimal submanifolds as the vanishing of the push-forward on the cotangent lift of a power of the symplectic form. The main theorem will then quickly follow by noting the Hamiltonian field spans the kernel of the symplectic form restricted to the cotangent lift.

Even in Riemannian geometry generic manifolds do not contain totally geodesic submanifolds. Nevertheless they exist and we end this paper giving two constructions of totally geodesic and hence minimal submanifolds in Finsler geometry. The first already appeared in [2] where the authors introduced the concept of isometric submersions for Finsler manifolds. Using it they gave examples of metrics on $\mathbb{C}P^n$ and $\mathbb{H}P^n$ for which all the geodesics are circles. We review this construction before proving the projective subspaces are furthermore totally geodesic. The second deals with the Grassmannian of oriented 2-planes of $\mathbb{R}^4$ as a symmetric space. Using the fact that the isotropy subgroup does not act transitively on the tangent space (contrarily to the case of the sphere) we are able to produce non-Riemannian Finsler metrics on this Grassmannian turning it into a symmetric space. However as Busemann already noticed in [10], Finsler symmetric spaces do not differ too much from Riemannian ones since they have the same parameterized geodesics. This remark helps us to easily find the geodesics of this symmetric space and to characterize its 2-dimensional totally geodesic submanifolds.

I am particularly gratefull to Juan Carlos Álvarez for all the interesting discussions we had and all the suggestions he made when writing this paper as well as for introducing me to this problem. I also would like to thanks Bruno Colbois for its invitation to the Math. department of Neuchâtel where this paper has partially been written.

2 Minimality of totally geodesic submanifolds

2.1 Finsler manifold

A *Finsler manifold* $(M^n, \|\cdot\|)$ is a manifold endowed with a smooth norm on each tangent space that varies smoothly with the base point. One does not require the norm to come from an inner product which makes the main (and, despite Riemann's thought, deeply conceptual) difference with Riemannian geometry. However we will impose upon the norm the following usual strict convexity property on each tangent space which ensures the dual norm to be smooth too (see [1]).

Definition 1 A smooth norm is Minkowski if, for every non-zero vector $v \in \overset{\circ}{T}_x M$, the quadratic form $\mathscr{Q}_v$ on the tangent space $T_x M$ defined by

$$\mathscr{Q}_v(w) = \frac{d^2}{dt^2}\left(\frac{1}{2}\|v + tw\|_x^2\right)_{|t=0}$$

is positive definite.

Geometrically, if v is a unit vector at x for a Minkowski norm, the hypersurface $\mathscr{E}_v$ of $T_x M$ defined by

$$\mathscr{E}_v \equiv \mathscr{Q}_v(w) = 1$$

is an ellipsoid centered at 0 which has a second order contact at v with the unit sphere of the norm. Hence one may characterize a Minkowski norm as a smooth norm whose unit sphere has everywhere a positive Gaussian curvature; its unit ball being consequently strictly convex.

While this geometrical viewpoint gives an easy to state and easy to picture characterization of a Minkowski norm, it is often more convenient in practice to take up a functional standpoint. This is classically done by introducing the Legendre map.

Definition 2 For every fixed $x \in M$, the Legendre map at x

$$\mathscr{L}_x : \overset{\circ}{T}_x M \to \overset{\circ}{T}{}^*_x M$$

is defined by

$$\mathscr{L}_x(v) \cdot w = \frac{d}{dt}\left(\frac{1}{2}\|v + tw\|_x^2\right)_{|t=0}.$$

One easily checks that this map is smooth and homogeneous of degree one. Hence it is completely characterized by the images of unit vectors and these are easy to picture. Indeed if v is a unit vector, then $\mathscr{L}_x(v)$ is the *unique* covector such that

$$\mathscr{L}_x(v) \cdot w = 1$$

for all vector w of the tangent hyperplane to the unit sphere at v. Also, recalling a norm is a convex function, it easily follows from this that the Legendre map is a (non linear) isometry between the tangent and cotangent spaces:

$$\|v\|_x = \|\mathscr{L}_x(v)\|_x^*$$

or equivalently that it maps the unit sphere onto the unit co-sphere. This naturally leads to the following functional characterization of a Minkowski norm. This Proposition is common knowledge and we will left its easy proof to the reader.

Proposition 1 *A smooth norm on $T_x M$ is Minkowski if and only if the Legendre map restricts to a diffeomorphism between the unit sphere of the tangent space $T_x M$ and the unit co-sphere in $T_x^* M$.*

Having a norm on each tangent space enables to define the length of (piecewise) smooth curves as the integral of the norm of the velocity vector:

$$L(\gamma[a, b]) = \int_a^b \|\dot{\gamma}(t)\| dt.$$

This in turn endows the Finsler manifold with a distance function making it a metric space:

$$d(x, y) = \inf\{L(\gamma[a, b]) \text{ s.t. } \gamma(a) = x \quad \text{and} \quad \gamma(b) = y\}.$$

Classically a curve that locally minimizes the distance between two of its points is called a *geodesic*. Moreover, the strict convexity property imposed upon the norm is equivalent to the ellipticity of the variational problem. It follows that every geodesic is a smooth curve uniquely defined by any close pair of its points just as in Riemannian geometry.

2.2 Hamiltonian viewpoint

It is well known that the 1-dimensional variational problem of finding the geodesics of a Finsler manifold is made a lot easier by using the symplectic geometry of the cotangent space.

In a few words, let

$$\rho\colon T^*M \to M \quad \text{and} \quad H = \frac{1}{2}\|\cdot\|^{*2}$$

be the cotangent bundle and the Hamiltonian function on the cotangent space respectively. Note that the Hamiltonian is smooth away from the zero section since the norm is Minkowski.

Let's now denote by α_M and $\omega_M = -d\alpha_M$ the canonical 1-form and the canonical symplectic 2-form respectively on the cotangent space, we will simply write α and ω when there will be no doubt about the manifold. The non-degeneracy of ω ensures the existence of a unique vector field X on T^*M, called the Hamiltonian vector field, such that

$$dH = \omega(X, \cdot). \tag{1}$$

One easily gets from this equation that this field is tangent to level sets of the Hamiltonian, or equivalently to level sets of the dual norm. In particular it is tangent to the unit co-sphere bundle S^*M.

Moreover it is well known that the geodesics of M are the projections of the integral curves of the Hamiltonian vector field:

Proposition 2 *The integral curves of the Hamiltonian vector field are those curves*

$$\gamma(t) = (x(t), q(t))$$

where $x(t)$ is a geodesic on M parameterized with constant speed and, for all t,

$$q(t) = \mathscr{L}_{x(t)}(\dot{x}(t)).$$

To stress the relationship between the 1-dimensional and the multi-dimensional variational problems, we give another characterization of the geodesics closer to our approach of minimal submanifolds. Consider a curve $x(t)$ on M parameterized with constant unit speed. We will call *cotangent lift of* $x(t)$ the following curve on the unit co-sphere bundle

$$\gamma(t) = (x(t), \mathscr{L}_{x(t)}(\dot{x}(t))).$$

Obviously the initial curve is a geodesic provide its cotangent lift is an integral curve of the Hamiltonian vector field. Using the non-degeneracy of ω and Eq. 1 this may be easily characterized as follows:

Proposition 3 *A curve $x(t)$ on M parameterized with constant unit speed is a geodesic if and only if at every point of its cotangent lift*

$$\omega(\dot{\gamma}(t), \cdot)_{|S^*M} = 0.$$

2.3 Holmes-Thompson volume

Let's now define the Holmes-Thompson volume of a Finsler manifold. The introduction of this volume was originally motivated by purely geometrical ideas about Minkowski spaces (see [14]) and its first definition is quite different from the symplectic one we will use (see [5] for the equivalence between both definitions). However this last tends to become the standard definition; furthermore we will gain from it a clearer understanding of the relationship between the minimality of curves for the length functional and the minimality of submanifolds for the Holmes-Thompson k-volume functional.

Since the symplectic form ω is non-degenerate its top order exterior power does not vanish and thus defines a volume form on the cotangent space which, divided by $n!$, is called the Liouville volume form:

$$\Omega = \frac{1}{n!}\omega^n.$$

We will use it to define the Holmes-Thompson volume:

Definition 3 The Holmes-Thompson volume of a n-dimensional Finsler manifold is the Liouville volume of its unit co-disc bundle divided by the Euclidean volume of the n-dimensional Euclidean unit ball.

Using Stokes theorem one may equivalently define the Holmes-Thompson as the volume of the unit co-sphere bundle:

$$\mathrm{Vol}^{HT}(M) = \frac{(-1)^{1+[n/2]}}{n!\,\epsilon_n} \int\limits_{S^*M} \alpha \wedge \omega^{n-1}$$

where ϵ_n is the Euclidean volume of the n-dimensional Euclidean unit ball (the sign is chosen to fit with the canonical orientation of the unit co-sphere bundle).

2.4 Geometry of submanifolds

An embedded submanifold $N \subset M$ inherits a Finsler structure by restriction of the Finsler norm to its tangent space. Hence one may consider the canonical symplectic geometry of its own cotangent space, let's call α_N and ω_N the corresponding forms, and use it to find the geodesics and to compute the Holmes-Thompson volume of this Finsler submanifold. However this is inadequate to study the extrinsic geometry of the submanifolds and irrelevant to compare the Holmes-Thompson volumes of close submanifolds or to characterize totally geodesic submanifolds; two necessary 'exercises' for our theorem. To overcome this, we will embed the unit co-sphere bundles of these submanifolds in the unit co-sphere bundle of the ambient Finsler manifold as in [6, Prop. 2.3]. It will turn out that this imbedding relates the symplectic geometry of the submanifold to the one of the ambient Finsler manifold, or more precisely that the pull-back of the canonical 1-form α_M by this embedding is α_N. Consequently this embedding will enable to study this part of the extrinsic geometry of submanifolds that is directly related to the symplectic geometry of the cotangent space and in particular to solve our two problems (see Prop. 5, 6).

Let's first picture geometrically the construction. Given a fixed point $x \in N$, the tangent space $T_x N$ is naturally embedded in $T_x M$,

$$i_x : T_x N \to T_x M.$$

Dually, one has a natural projection, defined by restriction,

$$\pi_x : T_x^* M \to T_x^* N.$$

This projection maps the unit ball, and even the unit sphere, of $T_x^* M$ onto the unit ball of $T_x^* N$. Moreover it follows from the strict convexity property of Minkowski norms that the unit sphere of $T_x^* N$ is diffeomorph to Σ_x: the singular set of the projection π_x restricted to the unit sphere of $T_x^* M$. This naturally leads to embed the unit co-sphere bundle $S^* N$ as

$$S^* N \hookrightarrow \bigcup_{x \in N} \Sigma_x \subset S^* M.$$

As for the characterization of Minkowski norms, it will be worthwhile to replace this geometric description by a functional standpoint using the Legendre map.

Proposition 4 *Let*

$$\mathscr{L}_N : \overset{\circ}{T}_x N \to \overset{\circ}{T}{}_x^* N \quad \text{and} \quad \mathscr{L}_M : \overset{\circ}{T}_x M \to \overset{\circ}{T}{}_x^* M$$

be the Legendre maps at $x \in M$. Then

$$\pi_x \circ \mathscr{L}_M = \mathscr{L}_N \tag{2}$$

and $\mathscr{L}_M \circ \mathscr{L}_N^{-1}$ restricts to a diffeomorphism from the unit sphere of T_x^*N to the singular set $\Sigma_x \subset T_x^*M$.

Proof From the definition of the Legendre map, one obviously gets that, for all non-zero vector $v \in T_xN$, the restriction of $\mathscr{L}_M(v)$ to T_xN is equal to $\mathscr{L}_N(v)$. Hence the Eq. 2. Moreover we know from Proposition 1 that $\mathscr{L}_M \circ \mathscr{L}_N^{-1}$ maps diffeomorphically the unit sphere of T_x^*N onto a submanifold of the unit sphere of T_x^*M. This submanifold must be contained in the singular set of the projection since it projects onto the boundary of the unit ball (see Eq. 2). Finally the singular set may not contain other points since for Minkowski norms the unit ball and its dual are strictly convex. □

Definition 4 The cotangent lift of the submanifold $N \subset M$ is the submanifold $\mathscr{N} \subset S^*M$ image of the unit co-sphere bundle S^*N by the smooth map

$$\begin{aligned} j \ : \ & S^*N \to S^*M \\ & (x, q) \mapsto (x, \mathscr{L}_M \circ \mathscr{L}_N^{-1}(q)) \end{aligned}$$

We may now solve our two 'exercises'. Indeed the cotangent lift of a submanifold varies obviously smoothly with the submanifold. Hence, the second equation of the following Proposition enables us to study the variations of the Holmes-Thompson volume of a submanifold under smooth perturbations using only the symplectic geometry of the ambient Finsler manifold. This Proposition is an immediate consequence of the Eq. 2.

Proposition 5 *Let $N^k \subset M$ be an embedded submanifold, then*

$$j^*\alpha_M = \alpha_N$$

and consequently

$$\mathrm{Vol}^{HT}(N) = \frac{(-1)^{1+[k/2]}}{k!\,\epsilon_k} \int\limits_{\mathscr{N}} \alpha_M \wedge \omega_M^{k-1}.$$

Furthermore totally geodesic submanifolds can also be characterized by means of the cotangent lift.

Proposition 6 *An embedded submanifold $N \subset M$ is totally geodesic if and only if the Hamiltonian vector field of M is tangent to the cotangent lift of N.*

Proof Recall that, for any Finsler manifold, a curve $\gamma(t) = (x(t), q(t))$ on the unit co-sphere bundle is an integral curve of the Hamiltonian vector field if and only if $x(t)$ is a geodesic parameterized with constant unit speed and $q(t) = \mathscr{L}_M(\dot{x}(t))$. The Proposition follows since $\dot{x}(t) = \mathscr{L}_N^{-1}(p(t))$ for some $p(t) \in T_{x(t)}^*N$. □

2.5 Minimal submanifolds

We have seen in Proposition 3 that the geodesics of a Finsler manifold are characterized by the vanishing of the symplectic form ω on their cotangent lift. This is extended to k-dimensional minimal submanifolds in Theorem 1 where we prove that these submanifolds are characterized by the nullity of the average of ω^k on their cotangent lifts; or more precisely by the nullity of the push-forward of this form along the fibers of the restricted bundle

$$\rho_N : \mathcal{N} \longrightarrow N.$$

However one should note that $\mathcal{N}$ is only $2k-1$-dimensional while we would like to integrate the $2k$-form ω^k. Hence we need to adapt the classical definition of the push-forward. We give first this definition, proving afterwards its consistency.

Definition 5 Let N be a k-dimensional submanifold of M and $\mathcal{N}$ its cotangent lift in S^*M. The push-forward of ω^k along the fibers of the restricted bundle $\rho_N : \mathcal{N} \to N$ is the form

$$(\rho_N \omega^k)_x \in T_x^*M \otimes \Lambda^k T_x^* N$$

defined by

$$(\rho_N \omega^k)_x(v, \mathbf{a}) = \int_{j(x)} \omega^k(\tilde{v}, \tilde{\mathbf{a}})$$

where $\tilde{v}$ is any lift of v in TS^*M and $\tilde{\mathbf{a}}$ is any lift of $\mathbf{a}$ in $\Lambda^k T\mathcal{N}$.

While it is clear as in every classical push-forward that $(\rho_N \omega^k)_x(v, \mathbf{a})$ will not depend on the choice of the lift $\tilde{\mathbf{a}}$, we need to prove it is also independent of the choice of the lift $\tilde{v}$. This is an easy consequence of the following lemma since it ensures $\omega^k(V, \cdot) = 0$ for any vector V tangent to the fiber of $\rho : S^*M \to N$.

Lemma 1 *Let (x, q) be a point of S^*M and V, W two tangent vectors based at this point such that*

$$D\rho(V) = 0 \quad \textit{and} \quad D\rho(W) = \mathcal{L}_N^{-1}(q).$$

Then

$$\omega(V, W) = 0.$$

Proof One may easily extend these two vectors into two commuting vector fields on S^*M satisfying the same two equalities. Moreover since q is a unit covector the second equality implies that $\alpha(W) = q(D\rho W) = 1$. Then

$$d\alpha(V, W) = V\alpha(W) - W\alpha(V) = 0$$

since $\alpha(V) = 0$ and $\alpha(W) = 1$. $\square$

Let's now characterize minimal submanifolds.

Theorem 1 *An embedded k-dimensional submanifold $N \subset M$ is minimal if and only if*

$$\rho_N \omega^k = 0.$$

Our proof is based on a direct computation of the first derivative of the Holmes-Thompson volume of a submanifold under small smooth variations. But a smooth variation of a submanifold does not directly give, via its differential, a smooth variation of its cotangent lift. One needs a scaling in each direction at each time. To avoid this technical annoyance we will rather work on the oriented projective bundle

$$P^+M = \overset{\circ}{T}{}^*M/\mathbb{R}^+.$$

This one is canonically diffeomorph to the unit co-sphere bundle. Hence we won't change our notations and use α and ω for the pull-back by this diffeomorphism of the canonical 1-form and the symplectic 2-form.

Proof of Theorem 1 Let N_t, $t \in]-\varepsilon, \varepsilon[$, be a compactly supported variation of a k-dimensional submanifold $N = N_0$ in a Finsler manifold $(M, \|\cdot\|)$. The differential of this variation defines a compactly supported variation $\mathscr{N}_t$ of the cotangent lift $\mathscr{N}$ in P^+M. Hence, up to a constant, the first variation of the Holmes-Thompson volume of N is given by

$$\frac{d}{dt}\int_{\mathscr{N}_t} \alpha \wedge \omega^{k-1}{}_{|t=0} = \int_{\mathscr{N}} L_V(\alpha \wedge \omega^{k-1})$$

where the Lie derivative is taken with respect to the (local) vector field V defined by the variation. Using Cartan's formula this is also equal to

$$\int_{\mathscr{N}} -i_V\omega^k + di_V\alpha \wedge \omega^{k-1}.$$

Hence by Stokes theorem the first variation of the volume of N is, up to a constant,

$$\int_{\mathscr{N}} i_V\omega^k.$$

Thus N is a minimal submanifold if and only if this quantity is zero for all compactly supported variation, hence if and only the push-forward of ω^k vanishes. □

Our main theorem now follows easily. As we will see in the proof, being totally geodesic already implies the form ω^k to vanish on the cotangent lift. Hence such submanifolds are minimal.

Theorem 2 *Totally geodesic submanifolds of a Finsler manifold are minimal for the Holmes-Thompson volume.*

Proof From the Proposition 6 we know that the Hamiltonian vector field X is tangent to the cotangent lift $\mathscr{N} \subset P^+M$. But the Eq. 1 tells us that the restriction of the 1-form $\omega(X, \cdot)$ to the unit co-sphere bundle must vanish. Equivalently

$$\omega(X, \cdot)_{|P^+M} = 0.$$

It follows that

$$\rho_N \omega^k = 0$$

and this proves the theorem. □

3 Examples of totally geodesic submanifolds

Even in Riemannian geometry, generic manifolds do not contain totally geodesic submanifolds. The existence of such submanifolds is in itself a strong condition on the curvature of the ambient manifold. However this does not prevent the existence of non-Riemannian Finsler examples and we will give two of them. We will work backward in both examples : the family of geodesics will be (almost) prescribed guaranteeing the existence of totally geodesic submanifolds and we will then construct suitable Finsler metrics.

3.1 Totally geodesic $\mathbb{C}P^k$ and $\mathbb{H}P^k$

3.1.1 Isometric submersions

In [2] the authors extended to Finsler manifolds the notion of isometric submersion. This enabled them to construct examples of Finsler metrics on $\mathbb{C}P^n$ and $\mathbb{H}P^n$ whose geodesics are all circles. We will quickly review their construction since, while they did not mention it, all the $\mathbb{C}P^k$ or $\mathbb{H}P^k$ are totally geodesic for these metrics (Fig. 1).

Definition 6 A smooth surjective map $\pi : M \to N$ between two Finsler manifolds is an isometric submersion if the differential $D\pi$ maps the unit balls of TM exactly onto the unit balls of TN.

Note that we have already encounters a particular example of an isometric submersion: the linear projection of the dual of a Minkowski space onto the dual of one of its subspace (see Fig. 2).

It easily follows from this definition that the projection π decreases the norm of tangent vectors and consequently the length of curves. However some unit tangent vectors of M are sent to unit tangent vectors of N, these are precisely the vectors of the singular set of the projection from the unit sphere to the unit ball

$$D\pi : S \to B.$$

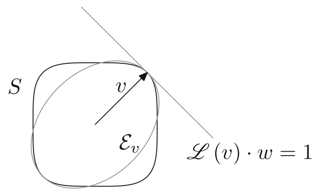

Fig. 1 Osculating ellipsoid and tangent hyperplane

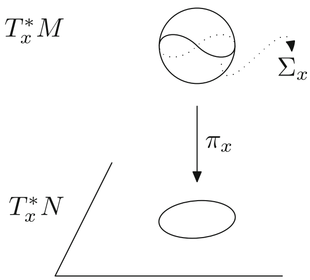

Fig. 2 Dual projection

These vectors and their multiples will be called *horizontal vectors*. One should note that the set of horizontal vectors at one point is a tangent *cone* and generally *not* a tangent subspace as it would be for Riemannian submersions (see Fig. 2).

Obviously a horizontal curve (i.e. all whose tangent vectors are horizontal) has the same length as its projection. This naturally leads to the following theorem.

Theorem 3 (Álvarez-Durán) *The geodesics of N are the projections of horizontal geodesics of M.*

3.1.2 Hopf fibration

The circle S^1 naturally acts on the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by multiplication with a unit complex number, the quotient space being obviously $\mathbb{C}P^n$. The projection

$$\begin{array}{l} S^{2n+1} \subset \mathbb{C}^{n+1} \\ \pi \downarrow \\ \mathbb{C}P^n \simeq S^{2n+1}/S^1 \end{array}$$

known as the *Hopf fibration* will be our base framework for constructing isometric submersions. This obviously extends to quaternionic projective spaces via the action of S^3 on $S^{4n+3} \subset \mathbb{H}^{n+1}$. However we will only develop the first case since the constructions and results would be exactly the same for the second, simply replacing the word *complex* by *quaternionic*.

We now use Busemann's construction [2,11] to produce Finsler metrics on the sphere S^{2n+1} having the great circles as geodesics. Assume ϕ to be a positive smooth measure on the manifold S^* of great hyperspheres of S^{2n+1} and define the length of a curve as the measure of the set of hyperspheres intersecting it:

$$L(\gamma) = \int_{\mathscr{S} \in S^*} \#(\mathscr{S} \cap \gamma)\phi.$$

The great circles necessarily are the geodesics of such a length structure since every hypersphere intersecting an arc of great circle also intersect any curve joining the ends points of this arc. Moreover Busemann proved there exists a Finsler metric on the sphere inducing the same length.

One may obtain a Finsler metric invariant under the action of the circle by asking the same invariance for the measure ϕ. In this case there exists a unique Finsler metric on $\mathbb{C}P^n$ for which the Hopf fibration is an isometric submersion. Indeed all the tangent unit balls based on points of $\pi^{-1}(x)$ project on the same centrally symmetric convex body of $T_x\mathbb{C}P^n$, this convex body being the unit ball of the norm on the tangent space of $\mathbb{C}P^n$.

Proposition 7 *For the Finsler metrics on $\mathbb{C}P^n$ described above, all the projective subspaces are totally geodesic.*

Proof We know from Theorem 3 that the geodesics of this Finsler metric on $\mathbb{C}P^n$ are the projections of some great circles of S^{2n+1} and according to [2] these projections are circles, but generally not great circles. Furthermore since every great circle of S^{2n+1} is contained in some complex 2-plane, its projection lies in a $\mathbb{C}P^1$. But 1-dimensional complex projective spaces are disjoint or transverse, hence they are all totally geodesic and consequently the $\mathbb{C}P^k$ too. □

3.2 Example of symmetric space

In [10] Busemann already noticed that Finsler symmetric spaces should not differ too much from Riemannian ones. Indeed he showed that topologically and "symmetrically" they are the same : one can always find a Riemannian metric on a Finsler symmetric space having the same involutive isometries and the same (parameterized) geodesics. However this does not imply that all symmetric spaces are necessarily Riemannian as metric spaces and we are going to give explicit examples of (non-Riemannian) Finsler metrics on the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ making it a Finsler symmetric space. It turns out that its geodesics and totally geodesic submanifolds are the same as for the classical Riemannian metric on this Grassmannian and we will review them in the last section.

Following Helgason [13],

Definition 7 A Riemannian (resp. Finsler) manifold M is a symmetric space if every point $p \in M$ is an isolated fixed point of an involutive isometry s_p of M.

Note that such an isometry s_p necessarily reverses the geodesics through p : if $\gamma(t)$ is a geodesic with $\gamma(0) = p$, then $s_p(\gamma(t)) = \gamma(-t)$.

As many symmetric spaces the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ is a homogeneous space with an invariant metric. Classically,

$$\mathscr{G} := G_2^+(\mathbb{R}^4) = SO(4)/SO(2) \times SO(2).$$

For simplicity, we will often write H for the isotropy subgroup $SO(2) \times SO(2)$ and also call π the projection

$$\pi : SO(4) \to \mathscr{G}.$$

Before constructing the Finsler metrics let's focus on the involutive isometries s_p. Classically these are obtained thanks to an involutive automorphism σ of the group $SO(4)$ fixing the isotropy subgroup [8, Theorem. 36.4]. We will consider the following:

$$\sigma(g) = h_0 \cdot g \cdot h_0 \quad \text{with} \quad h_0 = \begin{pmatrix} Id & 0 \\ 0 & -Id \end{pmatrix} \in SO(2) \times SO(2).$$

A straightforward computation shows that the isotropy subgroup is indeed the fixed point subgroup of this involution. Consequently σ induces an involution of the Grassmannian

$$\Sigma : \mathscr{G} \to \mathscr{G}$$

fixing the projection of the identity:

$$\Sigma(\pi(e)) = \pi(e).$$

All the involutive isometries will then be defined by

$$s_{\pi(g)} = g \cdot \Sigma \cdot g^{-1}.$$

3.2.1 Finsler metrics

We are now going to define metrics on the Grassmannian $\mathscr{G}$ invariant under the action of the group $SO(4)$. Obviously such metrics are completely characterized by Minkowski norms on the tangent space $T_{\pi(e)}\mathscr{G}$ which are invariant under the action of the isotropy subgroup. However it will be easier to define these metrics giving a Minkowski norm on some particular subspace $\mathfrak{k}$ of the Lie algebra $so(4)$. Moreover it will turn out (see Corollary 1) that the involutions s_p defined above are isometries for all these metrics turning the Grassmannian $\mathscr{G}$ into a symmetric space.

The differential of σ at the identity is a linear involutive automorphism of the Lie algebra $so(4)$. Hence this Lie algebra splits into the direct sum

$$so(4) = \mathfrak{h} \oplus \mathfrak{k}$$

where $\mathfrak{h}$, the Lie algebra of the isotropy subgroup H, is also the eigenspace of $D\sigma$ with eigenvalue 1 while $\mathfrak{k}$ is the eigenspace with eigenvalue –1.

Proposition 8 *The subspace $\mathfrak{k} \subset so(4)$ is invariant under the adjoint action of the isotropy subgroup H. Moreover this adjoint action on $\mathfrak{k}$ coincides with the left action of H on $T_{\pi(e)}\mathcal{G}$ via the linear isomorphism $D\pi : \mathfrak{k} \to T_{\pi(e)}\mathcal{G}$.*

Proof The invariance of $\mathfrak{k}$ under the adjoint action of H easily follows from the definition of the involution σ and the fact that h_0 commutes with every element of H.

Let X be a vector of the tangent space $T_{\pi(e)}\mathcal{G}$ and $\tilde{X}$ the unique vector of $\mathfrak{k}$ projecting on X. Then all the $\tilde{X} \cdot h^{-1}$ for $h \in H$ also project on X. Let $h_1 \in H$, then all the $h_1 \cdot \tilde{X} \cdot h^{-1}$ project on $h_1 \cdot X$, the only one in $\mathfrak{k}$ being $h_1 \cdot \tilde{X} \cdot h_1^{-1}$. □

Corollary 1 *Every Minkowski norm on $\mathfrak{k}$ invariant under the adjoint action of the isotropy subgroup H characterizes a Finsler metric on the Grassmannian $\mathcal{G}$ invariant under the action of $SO(4)$. Moreover the involutions s_p are isometries for these metrics.*

Proof Since $h_0 = h_0^{-1}$ the involution σ corresponds to the adjoint action of $h_0 \in H$. Hence the metrics are invariant under the involutions s_p. □

Let's now construct non-Riemannian examples of such metrics, or equivalently non-Euclidean examples of Minkowski norms on $\mathfrak{k}$. As well known Euclidean norms are characterized by their symmetry groups : in a n-dimensional space these are the only norms whose symmetry groups are isomorphic to $O(n)$. But while our norms must be invariant under the adjoint action of the isotropy subgroup, it turns out that this action on $\mathfrak{k}$ is not irreducible (see Prop. 9). Hence the symmetry groups of our norms must only contain a proper subgroup of the orthogonal group. This will enable us to give explicit examples of invariant non-Euclidean Minkowki norms (see Prop. 10).

One easily gets that the eigenspace of $D\sigma$ with eigenvalue –1 is simply

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \mid A \in \mathbb{R}^{2\times 2} \right\} \subset so(4),$$

which we will naturally identify with $\mathbb{R}^{2\times 2}$. We summarize the properties of the adjoint action of the isotropy subgroup on $\mathfrak{k}$ in the following Proposition whose proof is a straightforward computation.

Proposition 9 *The adjoint action of $(R_\theta, R_\eta) \in SO(2) \times SO(2)$ on $A \in \mathfrak{k}$ is given by*

$$R_\eta \cdot A \cdot R_{-\theta}.$$

This action is not irreducible and the space $\mathfrak{k}$ splits into a direct sum of two invariant subspaces, each one containing the elements of one connected component of $O(2)$ and their multiples.

It remains to introduce invariant Minkowski norms on $\mathfrak{k}$.

Proposition 10 *For all positive reals α, β with $\alpha < \beta < 2\alpha$, the map*

$$\begin{aligned} \mathfrak{h} &\longrightarrow \mathbb{R}^+ \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \frac{\alpha(a+d)^2 + \alpha(b-c)^2 + \beta(a-d)^2 + \beta(b+c)^2}{\sqrt{a^2+b^2+c^2+d^2}} \end{aligned}$$

is a Minkowski norm on $\mathfrak{h}$ invariant under the adjoint action of the isotropy subgroup.

Proof These maps are Minkowski norms according to [7, Theorem 2.2]. Moreover the invariance under the adjoint action of isotropy subgroup is equivalent to the invariance under rotations of each invariant subspace and this last is obviously satisfied by these norms. □

3.2.2 Totally geodesic submanifolds

As already mentioned the Finsler symmetric spaces have exactly the same geodesics and totally geodesic submanifolds as their Riemannian counterparts. This is a consequence of the following theorem characterizing the geodesics of symmetric spaces [8, Chap. IV, Theorem 3.3, 3.6]. While this theorem is originally stated for Riemannian symmetric spaces its proof is based on purely group-theoretic technics which apply for Finsler metrics too.

Theorem 4 *Let G/H be a symmetric space and X a non-zero vector of $\mathfrak{h}$. Then the geodesic emanating from $\pi(e)$ with tangent vector $D\pi(X)$ is given by*

$$\gamma(t) = \pi(exp\ tX).$$

Moreover, let $\mathfrak{m}$ be a subspace of $\mathfrak{h}$, then the subspace $D\pi(\mathfrak{m}) \subset T_{\pi(e)}G/H$ is tangent to a totally geodesic submanifold if and only if

$$[X, [Y, Z]] \in \mathfrak{m} \quad \text{for all } X, Y, Z \in \mathfrak{m}.$$

We will use this theorem to characterize the 2-dimensional totally geodesic submanifolds of the Grassmannian $\mathcal{G}$ after having described it in a more geometrical way as a product of two well known symmetric spaces. For this we will embed this Grassmannian in the space of 2-vectors of $\mathbb{R}^4$ as the submanifold of unit simple 2-vectors:

$$\mathcal{G} = \{\mathbf{a} \in \Lambda_s^2\mathbb{R}^4 \mid \|\mathbf{a}\| = 1\}.$$

It is well known that the cone of simple 2-vectors of $\mathbb{R}^4$ is characterized by the equation $\mathbf{a} \wedge \mathbf{a} = 0$, or equivalently in the standard coordinates

$$a_1a_6 - a_2a_5 + a_3a_4 = 0.$$

Applying a simple orthonormal change of coordinates, one concludes that the Grassmannian $\mathcal{G}$ is the intersection of the two following quadrics:

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 1 \\ a_1^2 + a_2^2 + a_3^2 - a_4^2 - a_5^2 - a_6^2 = 0 \end{cases}$$

In particular this Grassmannian is diffeomorph to the product

$$S_+^2 \times S_-^2$$

where S_+^2 (resp. S_-^2) stands for the sphere with radius $\sqrt{2}^{-1}$ in the 3-dimensional $a_1a_2a_3$-subspace of $\Lambda^2\mathbb{R}^4$ (resp. $a_4a_5a_6$-subspace of $\Lambda^2\mathbb{R}^4$). We will call W_+ and W_- these two subspaces.

The action of $SO(4)$ on $\mathbb{R}^4$ naturally induces an action on $\Lambda^2\mathbb{R}^4$, i.e. a morphism

$$SO(4) \longrightarrow SO(\Lambda^2\mathbb{R}^4).$$

This action preserves the cone of simple 2-vectors and maps $S_+^2 \times S_-^2$ isometrically onto itself. Obviously this restricted action on the embedding of the Grassmannian coincides with the natural action of $SO(4)$ on the Grassmannian itself.

As showed in next Proposition the action of $SO(4)$ on $\Lambda^2\mathbb{R}^4$ is not irreducible, leaving us with a decomposition of $\Lambda^2\mathbb{R}^4$ into a direct sum of invariant subspaces and a factorization of the action. This decomposition will lead to the geometric description of the Grassmannian as a product of simple homogeneous spaces, helping us to find totally geodesic submanifolds.

Proposition 11 *The two subspaces W_+ and W_- are invariants under the action of $SO(4)$. Moreover $SO(4)$ acts on $\Lambda^2\mathbb{R}^4 = W_+ \oplus W_-$ as $SO(3) \times SO(3)$ and the involutive automorphism σ on $SO(4)$ coincides with the classical involution*

$$\tau(g) = i_0 \cdot g \cdot i_0 \quad \textit{where} \quad i_0 = \begin{pmatrix} 1 & 0 \\ 0 & -Id \end{pmatrix}$$

on both copies of $SO(3)$.

Proof Consider the image of $S_+^2 \times \{q\}$, $q \in S_-^2$, under the action of one element of $SO(4)$. Since the group $SO(4)$ maps $S_+^2 \times S_-^2$ isometrically onto itself, this image must be a sphere with radius $\sqrt{2}^{-1}$. Moreover because $SO(4)$ is arcwise connected this image must be homotopic equivalent to $S_+^2 \times \{q\}$ in $S_+^2 \times S_-^2$. Hence its projection to W_+ has to be S_+^2. Consequently, $S_+^2 \times \{q\}$ is mapped isometrically onto $S_+^2 \times \{r\}$ for some $r \in S_-^2$. The invariance of W_+ then follows by linearity since every vector $w \in W_+$ such that $w = w_1 - w_2$ for some $w_i \in S_+^2 \times \{q\}$ will be mapped into W_+. Moreover W_- will also be invariant since it is perpendicular to W_+.

From the general theory of linear representations we know that since $\Lambda^2\mathbb{R}^4$ splits into a direct sum of two invariant subspaces the action of $SO(4)$ on it is equivalent to the action of $G \times H$ with G and H being two groups acting on W_+ and W_- respectively. Since the action of $SO(4)$ on the Grassmannian is transitive, its restricted action on W_+ will be transitive too leaving us with an action equivalent to the one of $SO(3)$ or $O(3)$. We then conclude since $SO(4)$ is connected.

Finally the assertion about the involutive automorphism follows from an easy computation on matrices. □

As an immediate consequence, we have the corollary:

Corollary 2 *As homogeneous space the Grassmannian of oriented 2-planes in* $\mathbb{R}^4$ *is the product*:

$$\mathcal{G} = SO(3)/SO(2) \times SO(3)/SO(2).$$

Then follows the characterization of 2-dimensional totally geodesic submanifolds:

Corollary 3 *With the metrics defined above on the Grassmannian* $\mathcal{G}$, *the 2-dimensional totally geodesic submanifolds are the Riemannian "round" spheres* $S^2_+ \times \{q\}$ *and* $\{p\} \times S^2_-$ *with the great circles as geodesics and the generically Finsler tori obtained as product of two great circles, one lying on* $S^2_+ \times \{q\}$ *and the other on* $\{p\} \times S^2_-$.

Proof Since the involutive automorphism of $SO(4)$ gives rise to the classical involution of $SO(3)$ the spheres $S^2_+ \times \{q\}$ and $\{p\} \times S^2_-$ are necessarily Riemannian "round" spheres with the great circles as geodesics.

It is well known that on a product of Riemannian manifolds $M_1 \times M_2$ every surface obtained as a product of two geodesics, one lying on M_1 and the other on M_2, is totally geodesic. Hence the tori are totally geodesic since on symmetric spaces the geodesics and totally geodesic submanifolds do not depend on the metric as seen in Theorem 4. Moreover any 2-dimensional tangent subspace at (p, q) is tangent to such a torus or to one of those spheres. Hence these are the only totally geodesic submanifolds.

Finally since 2-dimensional tangent subspaces are generically tangent to such a torus if these were generically Riemannian, the metric on the Grassmannian itself would have been Riemannian. □

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